W09 – Random Walks

Howard Berg's book (RW in biology)
Nelson's "Physical models of living systems" ch7

- Molecules subject to thermal fluctuations
- Organells inside a cell
- Mossitos infesting a forest (Karl Pearson 1905)
- Dust particles in air (Einstein)
- Genetic drift
- Cell mutations
- Ecology (Animal telemetry)
- Bacteria swimming in liquid media

1D random walk (discrete space and time)

Random Walk ⟶ Diffusion

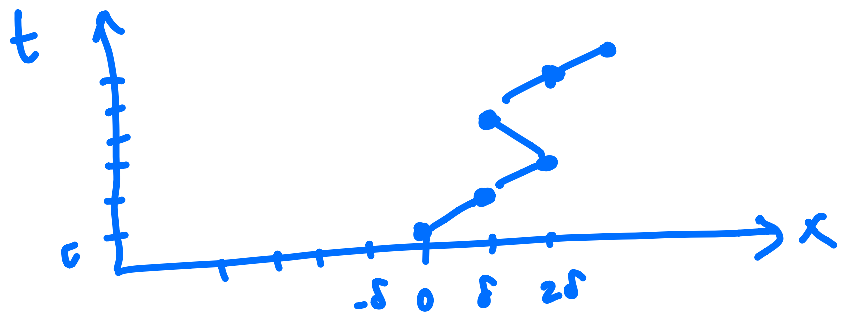
Random Walk	Diffusion
discrete time	Continuous time
" space	" space

1D Random Walks

How could a mosquito infest a forest (Pearson)

How dust moves in air (Einstein)

Start at $x=0$ $t=0$



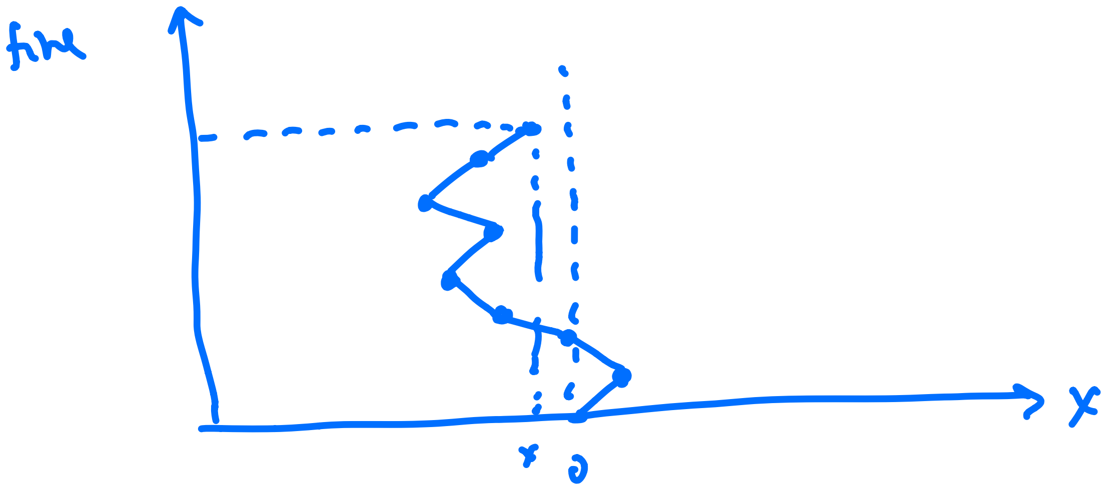
i) each particle moves after a fixed time "τ" a fixed step distance "δ" to the left or right.

ii) each particle steps
left with probability p
or right " " $q = 1-p$

iii) particles move independently from each other

iv) particles don't interact:
- they don't get destroyed
- they don't get created from other particles

The RW probability distribution



$P(x|t)$ probability particle is at position x after time t

$x = m \cdot \delta$ $\quad$ m = displacement in units of δ

$t = N \cdot \tau$ $\quad$ N = time steps

N steps, assume it takes ℓ = steps left

$N-\ell$ = steps right

$$P(\ell|N) = \binom{N}{\ell} p^{\ell} (1-p)^{N-\ell}$$

$$= \frac{N!}{\ell!\,(N-\ell)!}\, p^{\ell} (1-p)^{N-\ell}$$

Using the relationship t

$$m = \text{displacement} = (N-\ell)-\ell = N-2\ell$$

$$m = N-2\ell$$

$$\ell = \frac{N-m}{2}$$

$$N-\ell = N-\frac{N-m}{2} = \frac{N+m}{2}$$

$$\boxed{P(m|N) = \frac{N!}{\left(\frac{N-m}{2}\right)!\left(\frac{N+m}{2}\right)!}\, p^{\frac{N-m}{2}}(1-p)^{\frac{N+m}{2}}}$$

$\longrightarrow$ class code

Properties

$$\langle m \rangle = \sum_m m\, P(m|N) = \sum_\ell (N-2\ell)\, P(\ell|N)$$

$$= N\sum_\ell P(\ell|N) - 2\sum_\ell \ell\cdot P(\ell|N)$$

$$= N - 2\cdot Np = N(1-2p) = N(q-p)$$

Mean desplacement

$$\boxed{\langle m \rangle = N(q-p)}$$

if $p=q$ (Unbiased RW) $\langle m \rangle = 0$

- mean is proportional to # steps N
- mean is proportional to drift $q-p$

- Variance

$$Var(m) = Var(N-2\ell) = 4\,Var(\ell)$$

$$= 4 \cdot Npq$$

$$\sigma = \sqrt{Var(m)} = 2\sqrt{Npq}$$

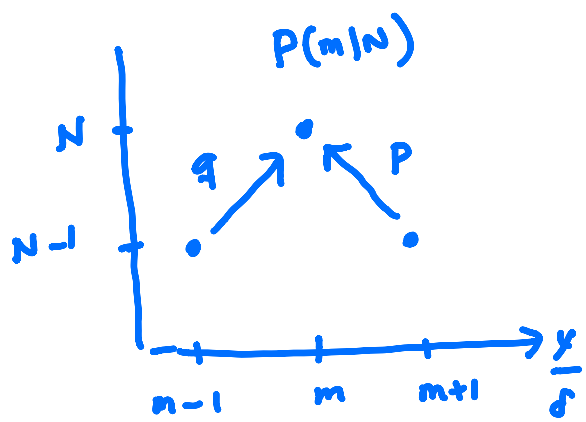
grows with $\sqrt{N}$

- In the limit
  $N \to \infty$, $Np \to \infty$ } p is finite and you take a very large # of steps

$P(\ell|N) \longrightarrow$ Normal distribution
$\mu = \langle m \rangle = N(q-p)$
$\sigma = 2\sqrt{Npq}$

$$P(m|N) \longrightarrow P(m|\mu,\sigma)\,dm = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(m-\mu)^2}{2\sigma^2}} \cdot dm$$

- RW has no memory — A Markov process



$$\boxed{P(m|N) = q\,P(m-1|N-1) + p \cdot P(m+1|N-1)}$$

Master equation of a diffusion stochastic process

Random walks w/ drunken pauses

$$p+q<1$$

$$1=p+q+z$$

$$\boxed{P(m|N) = q\,P(m-1|N-1) + p\,P(m+1|N-1) + z\,P(m|N-1)}$$

ℓ = # move left

n = # no move

$N-\ell-n$ = # move right

m = # right − # left

$= N-\ell-n-\ell = N-2\ell-n$

$$\boxed{\langle m\rangle} = N-2Np-Nz =$$

$$= N(1-2p-z)$$

$$= N(1-p-p-z)$$

$$= N(q+z-p-z) = \boxed{N(q-p)}$$

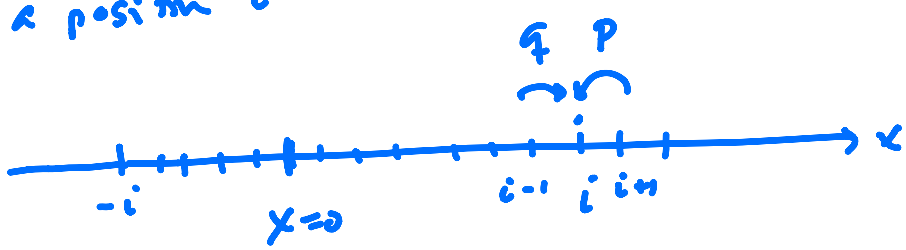
$$\mathrm{Var}(m) = 4\,\mathrm{Var}(d) + \mathrm{Var}(n)$$

$$= 4Np(1-p) + Nz(1-z)$$

$$= 4Np(q+z) + Nz(p+q)$$

Probability of Capture

A random walk starts at $x=0$, what is the probability that it reaches a position i



$P(\text{reaches } i \mid 0)$

- No risk of reaching the edges

$$P(+\infty \mid 0) = 0$$
$$P(-\infty \mid 0) = 0$$

- $$P(i \mid 0) = p\, P(i+1 \mid 0) + q\, P(i-1 \mid 0)$$

$$P(i \mid 0) = P(i-1 \mid 0) \cdot q + P(i+1 \mid 0) \cdot p$$

Solving for $P(0|0)$

Ansatz: $P(i|0) = A\beta^{i} + C$

$P(+\infty|0) = 0 \Rightarrow 0 < \beta < 1$

$P(-\infty|0) = 0 \qquad C = 0$

$$P(i|0) = A\beta^{i}$$

$P(i=0|0) = 1 = A$

$$P(i|0) = \beta^{i}$$

$$\beta^{i} = \beta^{i-1} q + \beta^{i+1} p$$

$$1 = \frac{q}{\beta} + \beta p$$

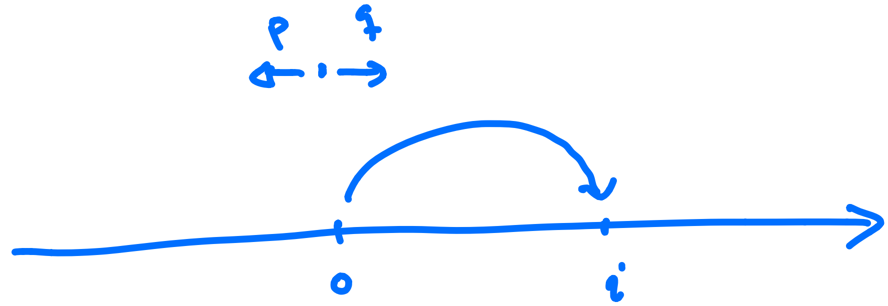
$$\beta = q + \beta^{2} p \quad , \quad \beta^{2} p - \beta + q = 0$$

$$\beta = \frac{1 \pm \sqrt{1 - 4pq}}{2p} = \frac{1 \pm \sqrt{(p-q)^2}}{2p}$$

$$= \frac{1 \pm (p-q)}{2p}$$

$$\beta = \frac{1 \pm (p-q)}{2p} \begin{cases} \dfrac{1+p-q}{2p} = \dfrac{2p}{2p} = 1 \\[2ex] \dfrac{1-p+q}{2p} = \dfrac{q}{p} \end{cases}$$

$$P(i|0) = \begin{cases} 1 & \text{if } q > p \\ \dfrac{q}{p} & \text{if } q < p \end{cases}$$



i) $q > 1$ always reaches $P(i|0) = 1$ if $q > p$

ii) if $q < p$ still there is a probability of reaching?

$$P(i|0) = \left(\frac{q}{p}\right)^i \quad \text{if } q < p$$

iii) $q = p$ $\quad P(i|0) = 1$ always reached (eventually)

Applications to ecology

a small population in which

q = prob of birth

$p = 1-q$ = prob of death

Even if $q < p$, there is a $\neq 0$ prob of reaching any population size i

$$\propto \left(\frac{q}{p}\right)^i$$

+ A gambler, no capital,

q = prob of winning $(q < p)$

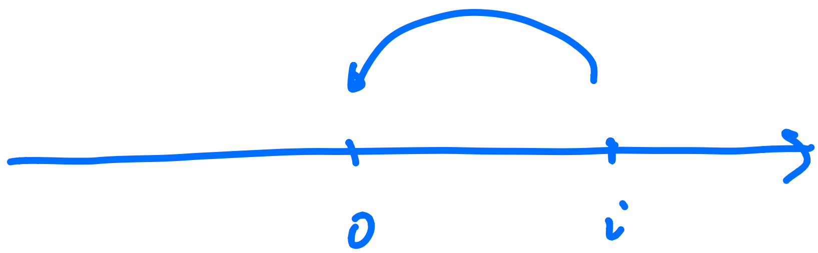
still can reach any gains i w/ prob

$$\left(\frac{q}{p}\right)^i$$

Reciprocally

$$P(0|i) = \begin{cases} \left(\frac{p}{q}\right)^i & p<q \\ 1 & p>q \end{cases}$$

$$q \leftarrow i \rightarrow p$$



Time to Capture (Hitting time)

How long it will take to reach a certain population size (or position)?

Introduce $h(j|i)$ as the expected time for a random walk to go from i to j

Using the Markov recursion

$$h(i|0) = 1 + h(i+1|0)\cdot p + h(i-1|0)\cdot q$$

Ansatz: $h(i/0) = Ai$

$$Ai = 1 + A(i+1)p + A(i-1)q$$

$$Ai = 1 + Ai(p+q) + Ap - Aq$$

$$Ai = 1 + Ai + Ap - Aq$$

$$0 = 1 - A(q-p)$$

$$A = \frac{1}{q-p}$$

$$h(i|0) = \begin{cases} \dfrac{i}{q-p} & i>0 \\ \dfrac{i}{p-q} & \text{if } i<0 \end{cases}$$

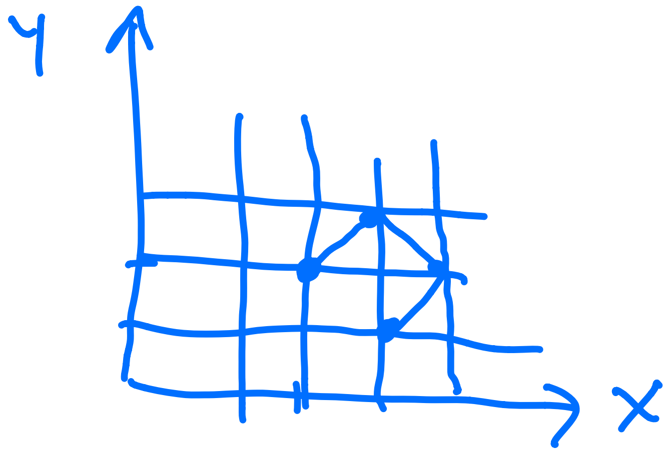
Notes

i) in general hitting times are not symmetric

ii) if $p=q$ $h(i|0) = +\infty$

iii) proportional to i

2D random walks



P_x = prob of moving left in X

P_y = prob of " down in Y

$$P(x,y\,|\,t+1) = p_x p_y\, P(x+1, y+1\,|\,t) + p_x q_y\, P(x+1, y-1\,|\,t) + q_x p_y\, P(x-1, y+1\,|\,t) + q_x q_y\, P(x-1, y-1\,|\,t)$$

- Particles tend to explore close regions
- There is no knowledge of what has not been explored yet.