Section 02 - Poisson distribution

MCB 111

September 16, 2022

The number of independent events in a time interval

An example from physical science: you are watching a meteor shower from Charles River. Suppose meteors arrive independently from each other, and during a very small time interval δt minute, the probability of seeing a meteor is proportional to the length of the time interval:

$$\mathbb{P}(\text{seeing a meteor during } \delta t \text{ minute}) \approx \lambda \delta t \tag{1}$$

How many stars (X) can I see for the entire meteor shower that lasts T minutes?

What is the probability that I see no meteor at all?

$$\begin{aligned}\mathbb{P}(X = 0) &\approx (1 - \lambda\delta t)^{\frac{T}{\delta t}} \\ &\approx \exp\left(-\lambda\delta t\frac{T}{\delta t}\right) \\ &= \exp\left(-\lambda T\right)\end{aligned} \tag{2}$$

What is the probability that I see exactly one meteor?

$$\begin{aligned}\mathbb{P}(X = 1) &\approx \binom{T/\delta t}{1} (\lambda \delta t)^1 (1 - \lambda \delta t)^{\frac{T}{\delta t} - 1} \\ &\approx \frac{T}{\delta t} (\lambda \delta t) \exp\left(-\lambda \delta t \frac{T}{\delta t} - \lambda \delta t\right) \\ &\approx \lambda T \exp\left(-\lambda T\right)\end{aligned} \tag{3}$$

What is the probability that I see exactly two meteors?

$$\begin{aligned}\mathbb{P}(X=1) &\approx \binom{T/\delta t}{2} (\lambda\delta t)^2 (1-\lambda\delta t)^{\frac{T}{\delta t}-2} \\ &\approx \frac{\frac{T}{\delta t}\left(\frac{T}{\delta t}-1\right)}{2!}(\lambda\delta t)^2 \exp\left(-\lambda\delta t\frac{T}{\delta t}-2\lambda\delta t\right) \\ &\approx \frac{(\lambda T)^2}{2!}\exp\left(-\lambda T\right)\end{aligned} \tag{4}$$

What is the probability that I see exactly n meteors?

$$\begin{aligned}
\mathbb{P}(X=n) &\approx \begin{pmatrix} T/\delta t \\ n \end{pmatrix} (\lambda\delta t)^n (1-\lambda\delta t)^{\frac{T}{\delta t}-n} \\
&\approx \frac{\frac{T}{\delta t}\cdots\left(\frac{T}{\delta t}-n+1\right)}{n!}(\lambda\delta t)^n \exp\left(-\lambda\delta t\frac{T}{\delta t}-n\lambda\delta t\right) \\
&\approx \frac{(\lambda T)^n}{n!}\exp\left(-\lambda T\right)
\end{aligned} \tag{5}$$

This formula gives us the Poisson distribution with parameter "λT"

Poisson distribution's probability mass function

$$\mathbb{P}(X = n) = \frac{\lambda^n}{n!} \exp(-\lambda), \quad n = 0, 1, 2, \cdots \tag{6}$$

Very useful when you are counting something that occurs independently in space/time with a fixed rate

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- Genome mutations

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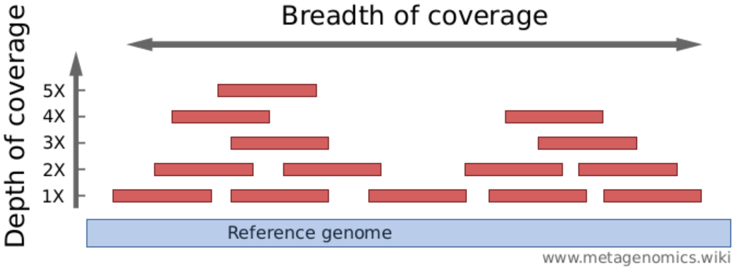
- Shotgun sequencing: the number of reads covering a site in the genome



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- The number of bird poops on the floor

Poisson distribution cannot be used to describe events with correlation among instances

Do you think these events in a given space/time can be described by a Poisson distribution?

- The number of heartbeats

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- The number of Canada geese in Harvard Yard (they move by families)
- The number of redline trains arriving at Harvard Sq

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- The number of fruitflies in BioLabs

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- The number of neurons firing in a neural network (cooperative firing)
- The number of fruitflies in BioLabs (Maybe, but correlation induced by experiments?)

Nice properties of the Poisson distribution

Parameter λ = Mean = Variance

$$
\begin{aligned}
\mathbb{E}(X) &= 0 + \sum_{n=1}^{\infty} n \exp(-\lambda) \frac{\lambda^n}{n!} \\
&= \exp(-\lambda) \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} \\
&= \lambda \exp(-\lambda) \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \\
&= \lambda \exp(-\lambda) \sum_{n=0}^{\infty} \frac{\lambda^n}{(n)!} \\
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&= \lambda
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"Variance=λ" is left for you to do

The sum of independent Poisson variables is also Poisson

$$\begin{aligned} & X_1 \sim \text{Poisson}(\lambda_1) \\ & X_2 \sim \text{Poisson}(\lambda_2) \\ & \Downarrow \\ & X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2) \end{aligned} \tag{8}$$

Why?

Intuitive answer:
In the meteor shower example, if λ_1 is the rate of seeing red meteors, and λ_2 is the rate of seeing green meteors, then in a short interval δ_t, the probability of seeing either a green or a red meteor is

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$$\begin{aligned} &\mathbb{P}(\text{seeing some meteor in } \delta t \text{ minutes}) \\ &\quad = 1 - \mathbb{P}(\text{seeing nothing}) \\ &\quad = 1 - (1 - \lambda_1 \delta t)(1 - \lambda_2 \delta t) \\ &\quad = (\lambda_1 + \lambda_2)\delta t - \lambda_1 \lambda_2 (\delta t)^2 \\ &\quad \approx (\lambda_1 + \lambda_2)\delta t \end{aligned} \tag{9}$$

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So you can define the rate of seeing either red or green meteors as $\lambda_1 + \lambda_2$, which defines another Poisson process (as well as a Poisson distribution).

This summation property is very useful in biology for developing null hypotheses where sub-processes of interest are independently Poisson. Some examples:

- The spike trains of some independent neurons
- Mutations at independent genomic regions
- DNA double-strand breaks across a group of cells

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In general, in a Poisson process with n types of outcomes, the probability that the next outcome is of type i is

$$\frac{\lambda_i}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \tag{10}$$