W03 - Model Comparison and hypothesis testing

- $P(D \mid H)$ $\qquad$ D = data
  H = hypothesis

- Compare
$$P(H_1 \mid D) \propto P(D \mid H_1)\, P(H_1)$$
$$P(H_2 \mid D) \propto P(D \mid H_2)\, P(H_2)$$

inference is based on data + (priors)

I? • Tomorrow Sep 22 is national vote registration date

I? • < 50% of Harvard students that can vote vote

• Harvard votes Challenge $\quad$ HVC
  www.voteschallenge.harvard.edu

Practical case Malaria Vaccine

- 40
  - 30 V ($N_V = 30$)
    - 1 had malaria (n_V)
    - 29 not ($N_V - n_V$)
  - 10 C ($N_c = 10$)
    - 3 malaria (n_c)
    - 7 not ($N_c - n_c$)

is V better than C ?

f_V = prob of contracting malaria in V treatment

f_c = " " " control (no treatment)

H_1 : V is better than C : $f_V < f_c$

H_0 : no difference btw V & C : $f_V = f_c$

- Classical χ^2 test
- Bayesian inference $P(H_1 | D)$, $P(H_2 | D)$

χ^2 test (chisquare, chi2)

* observed values $\{O_i\}_{i=1}^{N}$

* expected values $\{E_i\}_{i=1}^{N}$
  under null hypothesis (?)

* compare as $\chi^2_{val} = \sum_{i=1}^{N} \frac{(O_i - E_i)^2}{E_i}$

* Assume χ^2_{val} follows a χ^2 distribution
  with df degrees of freedom

  $df = N - \#$ constraints of problem

* Report p-value of χ^2_{val} under χ^2 distribution

$$\text{p-val}\left(\chi^2_{val}\right) = P_{\chi^2_{df}}\left(x \geq \chi^2_{val}\right)$$

		O	E
V_{30}		1	$30 \cdot f_{H_0} = \frac{30}{10} = 3$
		29	$30 \cdot (1-f_{H_0}) = 30 \cdot \frac{9}{10} = 27$
C_{10}		3	$10 \cdot f_{H_0} = \frac{10}{10} = 1$
		7	$10 \cdot (1-f_{H_0}) = 10 \cdot \frac{9}{10} = 9$

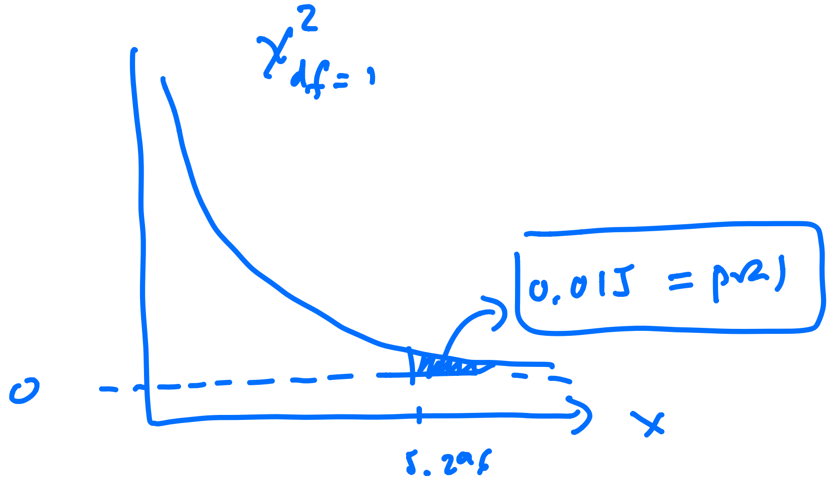
f_{H_0} = prob of contracting malaria under null hy

$$= \frac{1+3}{40} = \frac{1}{10}$$

$$\chi^2_{val} = \frac{(1-3)^2}{3} + \frac{(29-27)^2}{27} + \frac{(3-1)^2}{1} + \frac{(7-9)^2}{9}$$

$$= 5.926$$

$$df = 4 - 3 \begin{cases} V: 30 \\ C: 10 \\ f_{H_0} \end{cases} = 1$$



in classical stats

pval < 0.05 → reject the null hypothesis.

we reject $v = c$

Note

① $p = 0.015$ DOES NOT MEAN

that the prob that $v \neq c$ is 85%

② what it means:

if you were to repeat the same exp many times, and the data follows the null hypothesis, there would be a 1.5% chance of getting 5.926 or higher

~ very indirect result about whether 2 treatments have different effectiveness

③ we have not used the H_1 hypothesis only H_0

How does χ^2-test work?

Given df random variables that follow $N(0,1)$

the sum of the random variables follows

a χ^2_{df} distribution.

$$\frac{(O_i - E_i)^2}{E_i} \sim N(0,1)$$

can we test this statement?

how about $\sigma = 1$?

Which χ^2-test

- pearson's χ^2 test
- Yates χ^2 test

⋮

$$\chi^2_{Yates} = \sum_i \frac{(|O_i - E_i| - 0.05)^2}{E_i}$$

$$\chi^2_{Yates} = 3.34 \rightarrow pval \approx 0.7$$

Bayes or Chi-square or does it not matter?

Mackay

$$P(D \mid f_v f_c) = P(D_v \mid f_v)\, P(D_c \mid f_c)$$

$$P(D_v \mid f_v) = \binom{N_v}{n_v} f_v^{n_v} (1-f_v)^{N_v-n_v}$$

$$P(D_c \mid f_c) = \binom{N_c}{n_c} f_c^{n_c} (1-f_c)^{N_c-n_c}$$

$$P(f_c f_v \mid D) = \frac{P(D_v \mid f_v) \cdot P(D_c \mid f_c)\, P(f_v f_c)}{P(D)}$$

prior $P(f_v f_c) = P(f_v) \cdot P(f_c)$

$$P(f_c f_v \mid D) = \underbrace{\frac{P(D_v \mid f_v)\, P(f_v)}{P(D_v)}}_{P(f_v \mid D_v)} \cdot \underbrace{\frac{P(D_c \mid f_c)\, P(f_c)}{P(D_c)}}_{P(f_c \mid D_c)}$$

Sections last week

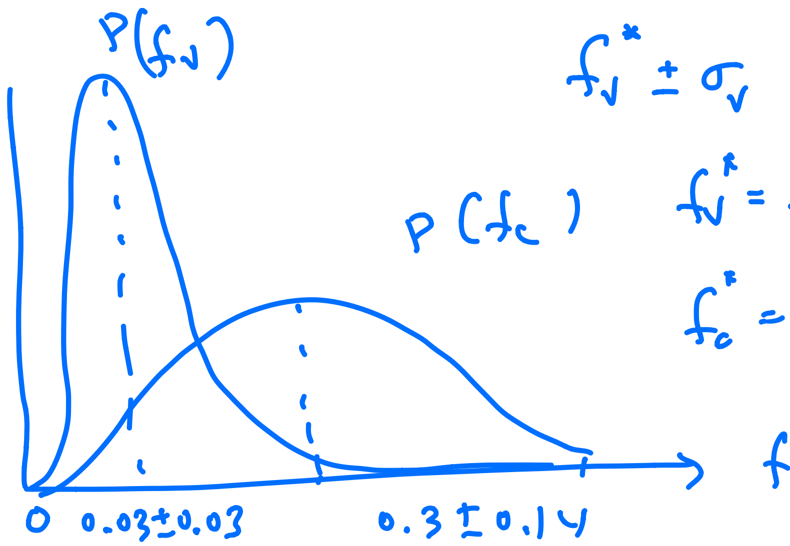
$$P(f_v \mid D_v) = \frac{\binom{N_v}{n_v} f_v^{n_v} (1-f_v)^{N_v-n_v}}{\binom{N_v}{n_v} \int_0^1 g^{n_v} (1-g)^{N_v-n_v}\, dg} \; P(f_v)$$

$P(f_v) = 1$

$$= \frac{f_v^{n_v} (1-f_v)^{N_v-n_v}}{\int_0^1 g^{n_v} (1-g)^{N_v-n_v}\, d g}$$

$$P(f_v \mid D_v) = \frac{(N_v+1)!}{n_v!\,(N_v-n_v)!} \; f_v^{n_v} (1-f_v)^{N_v-n_v}$$

$$P(f_c \mid D_c) = \frac{(N_c+1)!}{n_c!\,(N_c-n_c)!} \; f_c^{n_c} (1-f_c)^{N_c-n_c}$$



$f_v^* \pm \sigma_v$

$f_v^* = \frac{n_v}{N_v} = \frac{1}{30} = 0.03$

$f_c^* = \frac{n_c}{N_c} = \frac{3}{10} = 0.3$

$\boxed{H_1}$; $f_v < f_c$

$$P(f_v\, f_c | D) = P(f_v | D_v)\, P(f_c | D_c)$$

$$P(H_1 | D) = P(f_v < f_c | D)$$

$$= \int_0^1 df_c \int_0^{f_c} df_v \cdot P(f_v | D_v)\, P(f_c | D_c)$$

$$= \frac{(N_v+1)!}{n_v!\,(N_v-n_v)!} \; \frac{(N_c+1)!}{n_c!\,(N_c-n_c)!} \times$$

$$\times \int_0^1 df_c\, f_c^{n_c} (1-f_c)^{N_c-n_c} \int_0^{f_c} f_v^{n_v} (1-f_v)^{N_v-n_c}\, df_v$$

$$= 0.987$$