Bayesian Inference of parameters

Data: $t_1, \dots, t_6$ $\qquad P(t|\lambda) = \dfrac{e^{-t/\lambda}}{Z(\lambda)}$

$$P(\text{data}|\lambda) = P(t_1 \dots t_6|\lambda) = \prod_i P(t_i|\lambda) = \prod_i \frac{e^{-t_i/\lambda}}{Z(\lambda)}$$

independent events

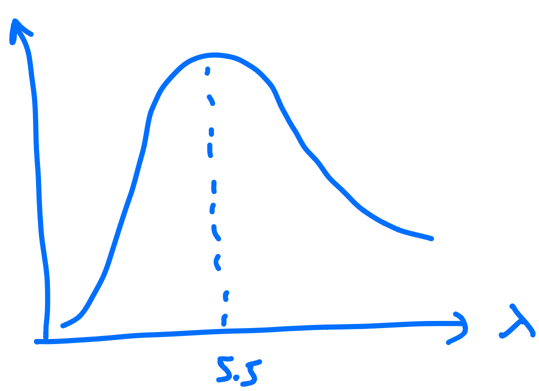
$$= \frac{e^{-\sum_i t_i/\lambda}}{Z^N(\lambda)}$$

Bayes theorem

$$P(\lambda|\text{data}) \propto P(\text{data}|\lambda) \cdot P(\lambda)$$

$P(\lambda)$ prior, not dependent λ

$$P(\lambda|\text{data}) \propto \frac{e^{-\sum_i t_i/\lambda}}{Z^N(\lambda)}$$



$[a,b] = [0,20]$

$$P(\lambda \mid \text{data}) = \frac{\overset{\text{likelihood}}{P(\text{data} \mid \lambda)} \cdot \overset{\text{prior}}{P(\lambda)}}{\underset{\text{evidence}}{P(\text{data})}}$$

$$\propto P(\text{data} \mid \lambda) \cdot P(\lambda)$$

"What we know about λ after the data arrives is what we knew before, $P(\lambda)$, and what the data told you $P(\text{data} \mid \lambda)$"

Mackay

- depends on assumptions $P(\lambda)$

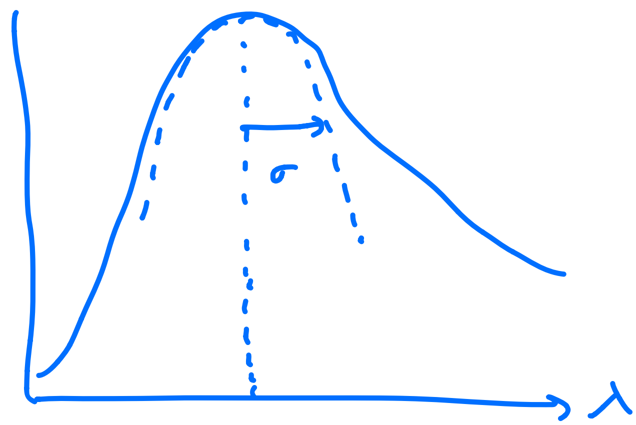
  "we cannot do inference w/o making assumptions"

- at least the assumptions are explicitly stated

→ python demonstration

Best estimates and confidence intervals

If forced to give an estimate of the parameter....



λ^* → maximum of $P(\lambda | data)$

$L(\lambda) = \log P(\lambda | data)$

max of $P(\lambda|data)$ same as max of $L(\lambda)$

$$L(\lambda) \simeq L(\lambda^*) + \left.\frac{dL}{d\lambda}\right|_{\lambda^*} (\lambda - \lambda^*) + \frac{1}{2} \left.\frac{d^2L}{d\lambda^2}\right|_{\lambda^*} (\lambda - \lambda^*)^2$$

$+ \ldots$

$$\lambda^* ; \quad \left.\frac{dL}{d\lambda}\right|_{\lambda = \lambda^*} = 0$$

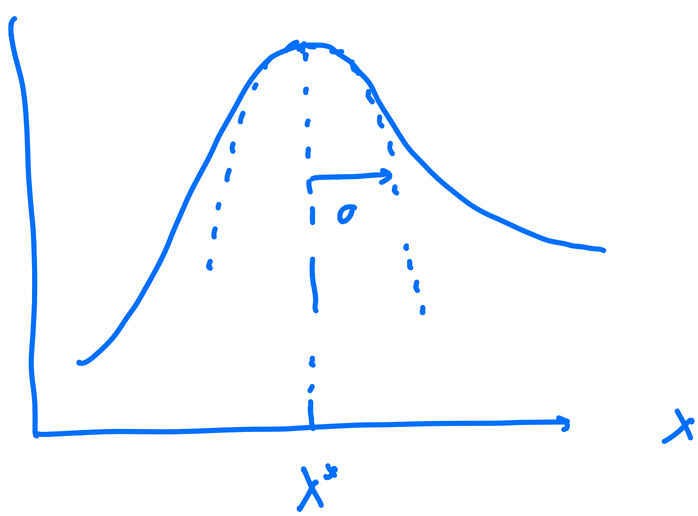
$$L(\lambda) = L(\lambda^*) + \frac{1}{2} \left.\frac{d^2L}{d\lambda^2}\right|_* (\lambda - \lambda^*)^2$$

$$P(\lambda | \text{data}) = e^{L(\lambda^*)} \, e^{\frac{1}{2} \left.\frac{d^2L}{d\lambda^2}\right|_* (\lambda - \lambda^*)^2}$$

$$\propto e^{-\frac{(\lambda-\lambda^*)^2}{2\sigma^2}}$$

$$-\frac{1}{\sigma^2} = \left.\frac{d^2L}{d\lambda^2}\right|_*$$

$$\lambda^* ; \left.\frac{dL}{d\lambda}\right|_* = 0 \quad \text{and} \quad \sigma^2 = -\frac{1}{\left.\frac{d^2L}{d\lambda^2}\right|_*}$$



Best estimate and confidence for the bacterial mutation wait times

$$P(\lambda | \text{data}) = A \frac{e^{-\sum_i t_i/\lambda}}{\lambda^N \left(e^{-a/\lambda} - e^{-b/\lambda}\right)^N} \qquad " \quad \hat{t} = \frac{1}{N} \sum_i t_i$$

$$\log P = \log A - \frac{N\hat{t}}{\lambda} - N \log \lambda - N \log\left(e^{-a/\lambda} - e^{-b/\lambda}\right)$$

$a = 0 \quad b \to \infty$

$\log(1 - 0) = 0$

$$L = \log P = \log A - \frac{N\hat{t}}{\lambda} - N \log \lambda$$

$$\frac{\delta L}{\delta \lambda} = \frac{N\hat{t}}{\lambda^2} - \frac{N}{\lambda} = 0 \qquad \frac{N\hat{t}}{\lambda_*^2} = \frac{N}{\lambda^*} \Rightarrow \frac{\hat{t}}{\lambda_*} = 1 \Rightarrow \boxed{\hat{\lambda} = \hat{t}}$$

$$\frac{\delta^2 L}{\delta \lambda^2} = -2\frac{N\hat{t}}{\lambda^3} + \frac{N}{\lambda^2} = \frac{N}{\lambda^2}\left[1 - \frac{2\hat{t}}{\lambda}\right]_{\lambda=\lambda^*} = \frac{N}{\hat{t}^2}\left[1 - \frac{2\hat{t}}{\hat{t}}\right] = -\frac{N}{\hat{t}^2}$$

$$\frac{dL^{-1}}{dL} = -L^{-2}$$

$$\frac{dL^{-2}}{dL} = -3L^{-3}$$

$$\frac{1}{\sigma^2} = \frac{N}{\hat{t}^2} \to \boxed{\sigma = \frac{\hat{t}}{\sqrt{N}}}$$

bayes estimate: $\boxed{\lambda \simeq \hat{t} \pm \frac{\hat{t}}{\sqrt{N}}}$